

Potentially Semi-stable Deformations of Specified Hodge-Tate type and Galois type

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Abstract

Let k be a perfect field of characteristic $p > 2$, and let K be a finite totally ramified extension of $W(k)[\frac{1}{p}]$. We prove that the locus of potentially semi-stable $\text{Gal}(\bar{K}/K)$ -representations of a given Hodge-Tate type and Galois type is a closed subspace of the universal deformation ring, generalizing the result of [Kis07] where k is assumed to be finite.

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1 Introduction

Let k be a perfect field of characteristic $p > 2$, and let $W(k)$ be its ring of Witt vectors. Write $K_0 = \text{Frac}(W(k))$, and let K/K_0 be a finite totally ramified extension. We fix an algebraic closure \bar{K} of K , and let $\mathcal{G}_K := \text{Gal}(\bar{K}/K)$ be the absolute Galois group of K .

Let E/\mathbf{Q}_p be a finite extension with residue field \mathbf{F} , and let V_0 be a finite dimensional \mathbf{F} -representation of \mathcal{G}_K . Denote by \mathcal{C} the category of local topological \mathcal{O}_E -algebras A such

that the natural map $\mathcal{O}_E \rightarrow A/\mathfrak{m}_A$ is surjective and the map from A to the projective limit of its discrete artinian quotients is a topological isomorphism. If V_0 is absolutely irreducible, then there exists a universal deformation ring $R \in \mathcal{C}$ with a deformation V_R which parametrizes the isomorphism classes of deformations of V_0 ([SL97]). Note that R is not necessarily noetherian in general when k is not finite.

In this paper, we study the geometry of the locus of potentially semi-stable representations with a specified Hodge-Tate type \mathbf{v} and Galois type τ . We show that such a locus cuts out a closed subspace in the following sense:

Theorem A. *There exists a closed ideal $\mathfrak{a}_{\mathbf{v},\tau} \subset R$ such that the following holds: for any finite flat \mathcal{O}_E -algebra A and a continuous \mathcal{O}_E -algebra homomorphism $f : R \rightarrow A$ (where we equip A with the (p) -adic topology), the induced representation $A[\frac{1}{p}] \otimes_{f,R} V_R$ is potentially semi-stable of Hodge-Tate type \mathbf{v} and Galois type τ if and only if f factors through the quotient $R/\mathfrak{a}_{\mathbf{v},\tau}$.*

When the residue field k is finite, Kisin proved the corresponding result in [Kis07, Theorem 2.7.6]. One of the main steps in [Kis07] is the construction of the projective scheme which parametrizes representations of $E(u)$ -height $\leq r$ for a fixed positive integer r (cf. [Kis07, Section 1.2]). It is obtained as a closed subscheme of the affine Grassmannian for the restriction of scalars $\mathrm{Res}_{W(k)/\mathbf{Z}_p} \mathrm{GL}_d$. But this construction does not make sense in general when k is infinite. The main difficulty is that we do not know how to analyze whether the restriction of scalars $\mathrm{Res}_{W(k)/\mathbf{Z}_p}$ for a non-affine scheme over $W(k)$ is representable by an Ind-scheme when k is infinite, even for simple examples such as $\mathbb{P}_{W(k)}^1$.

Another approach to studying the locus cut out by certain p -adic Hodge theoretic conditions, motivated by Fontaine's conjecture in [Fon97], is to analyze torsion representations given as the subquotients of Galois stable lattices satisfying the given conditions. For semi-stable (or crystalline) representations having Hodge-Tate weights in $[0, r]$, this is carried out by Liu in [Liu07]. And in the case k is finite, Liu proved the corresponding result for semi-stable representations of a given Hodge-Tate type in [Liu15].

To study the refined structure of a Hodge-Tate type and Galois type of torsion representations, we use the functor given in [Liu12] from the category of representations semi-stable over a totally ramified Galois extension K'/K to the category of lattices in filtered modules equipped with Frobenius, monodromy, and $\mathrm{Gal}(K'/K)$ -action. In this paper, we first generalize the method in [Liu15] to the case k is not necessarily finite, and study semi-stable representations of a given Hodge-Tate type. Then, we study potentially semi-stable representations of a given Galois type and show that those p -adic Hodge-theoretic conditions are p -adically closed (cf. Theorem 3.3 and 3.17).

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2 Torsion Representation and Construction of M_{st}

We keep the notations as in the introduction. Let $K' \subset \bar{K}$ be a finite totally ramified Galois extension of K . In this section, we will first explain the construction of the functor given in [Liu12] from the category of representations semi-stable over K' to the category of lattices in filtered modules equipped with Frobenius, monodromy, and $\text{Gal}(K'/K)$ -action. Then, we will explain the result proved in [Liu12] and [Liu15] that one can associate a Hodge-Tate type and Galois type to a torsion representation up to some constant depending only on K' .

2.1 Potentially Semi-stable Representation and Filtered (φ, N, Γ) -module

For a \mathcal{G}_K -representation V over \mathbf{Q}_p , we say V is *potentially semi-stable* if there exists a finite extension $L \subset \bar{K}$ of K such that V restricted to $\mathcal{G}_L := \text{Gal}(\bar{K}/L)$ is semi-stable. This means precisely that $\dim_{\mathbf{Q}_p} V = \dim_{L_0} (B_{\text{st}} \otimes_{\mathbf{Q}_p} V^\vee)^{\mathcal{G}_L}$ where L_0 is the maximal unramified subextension of L/K_0 .

Let $e' = [K' : K_0]$. We fix a uniformizer π of K' , and let $F(u)$ be the Eisenstein polynomial for π over K_0 . Denote by $\text{Rep}_{\mathbf{Q}_p}^{\text{pst}, K'}$ the category of \mathcal{G}_K -representations over \mathbf{Q}_p which become semi-stable over K' (i.e., semi-stable as $\text{Gal}(\bar{K}/K')$ -representations). Let $\Gamma = \text{Gal}(K'/K)$ and $\mathcal{G}_{K'} = \text{Gal}(\bar{K}/K')$. Note that K_0 is equipped with the natural Frobenius endomorphism φ .

We consider the category of filtered (φ, N, Γ) -modules whose objects are finite dimensional K_0 -vector spaces D equipped with:

- a Frobenius semi-linear injection $\varphi : D \rightarrow D$,
- $W(k)$ -linear map $N : D \rightarrow D$ such that $N\varphi = p\varphi N$,
- decreasing filtration $\text{Fil}^i D_{K'}$ on $D_{K'} := K' \otimes_{K_0} D$ by K' -sub-vector spaces such that $\text{Fil}^i D_{K'} = D_{K'}$ for $i \ll 0$ and $\text{Fil}^i D_{K'} = 0$ for $i \gg 0$, and
- K_0 -linear action by Γ on D which commutes with φ and N . If we extend Γ -action semi-linearly to $D_{K'}$, then for any $\gamma \in \Gamma$, $\gamma(\text{Fil}^i D_{K'}) \subset \text{Fil}^i D_{K'}$.

Morphisms between filtered (φ, N, Γ) -modules are K_0 -linear maps compatible with all structures. The functor $D_{\text{st}}^{K'} : V \mapsto (B_{\text{st}} \otimes_{\mathbf{Q}_p} V^\vee)^{\mathcal{G}_{K'}}$ is an equivalence between $\text{Rep}_{\mathbf{Q}_p}^{\text{pst}, K'}$ and the category of *weakly admissible* filtered (φ, N, Γ) -modules (cf. [CF00], [Fon94]).

We define an integral structure of a filtered (φ, N, Γ) -module.

Definition 2.1. Let D be a filtered (φ, N, Γ) -module. A *lattice* M in D is a finite free $W(k)$ -submodule of D such that $M[\frac{1}{p}] := M \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \cong D$, and $\varphi(M) \subset M$, $N(M) \subset M$, and $\gamma(M) \subset M$ for all $\gamma \in \Gamma$. For a lattice $M \subset D$, we equip $M_{K'} := \mathcal{O}_{K'} \otimes_{W(k)} M$ with the natural filtration by $\mathcal{O}_{K'}$ -submodules, given by $\text{Fil}^i M_{K'} = M_{K'} \cap \text{Fil}^i D_{K'}$. If M_1, M_2 are lattices in filtered (φ, N, Γ) -modules D_1, D_2 respectively, then a morphism $f : M_1 \rightarrow M_2$ is a $W(k)$ -linear map such that $f \otimes_{\mathbf{Z}_p} \mathbf{Q}_p : D_1 \rightarrow D_2$ is a morphism of filtered (φ, N, Γ) -modules.

Note that for a lattice M in a filtered (φ, N, Γ) -module, the associated graded $\mathcal{O}_{K'}$ -modules $\text{gr}^i M_{K'} = \text{Fil}^i M_{K'} / \text{Fil}^{i+1} M_{K'}$ is torsion free by the definition of the filtration.

Let r be a positive integer. Denote by $L^r(\varphi, N, \Gamma)$ the category of lattices in filtered (φ, N, Γ) -modules D satisfying $\text{Fil}^0 D_{K'} = D_{K'}$ and $\text{Fil}^{r+1} D_{K'} = 0$. Let $\text{Rep}_{\mathbf{Q}_p}^{\text{pst}, K', r}$ be the full subcategory of $\text{Rep}_{\mathbf{Q}_p}^{\text{pst}, K'}$ whose objects have Hodge-Tate weights in $[0, r]$, and let $\text{Rep}_{\mathbf{Z}_p}^{\text{pst}, K', r}$ be the category of \mathcal{G}_K -stable \mathbf{Z}_p -lattices of representations in $\text{Rep}_{\mathbf{Q}_p}^{\text{pst}, K', r}$. The following theorem is proved in [Liu12]:

Theorem 2.2. (cf. [Liu12, Theorem 2.3]) *There exists a faithful contravariant functor M_{st} from $\text{Rep}_{\mathbf{Z}_p}^{\text{pst}, K', r}$ to $L^r(\varphi, N, \Gamma)$. If we denote by $M_{\text{st}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ the functor M_{st} associated to the isogeny categories, then there exists a natural isomorphism of functors between $M_{\text{st}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $D_{\text{st}}^{K'}$.*

2.2 Construction of M_{st}

We now explain briefly the construction in [Liu12] of the functor M_{st} in Theorem 3.3. We first recall the definitions of period rings in p -adic Hodge theory.

Let S' be the p -adic completion of the divided power-envelope of $\mathfrak{S} = W(k)[[u]]$ with respect to the ideal $(F(u))$. Denote $S'_{K_0} := S'[\frac{1}{p}]$. Let C_p be the p -adic completion of \bar{K} , and let \mathcal{O}_{C_p} be its ring of integers. We define $\mathcal{O}_{C_p}^b := \varprojlim_{x \mapsto x^p} \mathcal{O}_{C_p}/p$. By the universal property of the ring of Witt vectors $W(\mathcal{O}_{C_p}^b)$, there exists a unique surjection $\theta : W(\mathcal{O}_{C_p}^b) \rightarrow \mathcal{O}_{C_p}$, which lifts the projection $\mathcal{O}_{C_p}^b \rightarrow \mathcal{O}_{C_p}/p$ onto the first factor of the inverse limit. We denote by B_{dR}^+ the $\ker(\theta)$ -adic completion of $W(\mathcal{O}_{C_p}^b)[\frac{1}{p}]$. Let A_{cris} be the p -adic completion of the divided power-envelope of $W(\mathcal{O}_{C_p}^b)$ with respect to $\ker(\theta)$. We fix a compatible system of p^n -th roots $\pi_n \in \mathcal{O}_{\bar{K}}$ of π for non-negative integers n , and let $\underline{\pi} := (\pi_n) \in \mathcal{O}_{C_p}^b$. We have an embedding $\mathfrak{S} \hookrightarrow W(\mathcal{O}_{C_p}^b)$ mapping u to $[\underline{\pi}]$, and hence the embeddings $\mathfrak{S} \hookrightarrow S' \hookrightarrow A_{\text{cris}}$ compatible with Frobenius endomorphisms. Let $B_{\text{cris}}^+ = A_{\text{cris}}[\frac{1}{p}]$. Let $\mathbf{u} = \log[\underline{\pi}]$, and

$B_{\text{st}}^+ = B_{\text{cris}}^+[\mathbf{u}]$. We also fix a compatible system of primitive p^n -th roots of unity $\zeta_{p^n} \in \mathcal{O}_{\bar{K}}$ for non-negative integers n , and let $\epsilon := (\zeta_{p^n}) \in \mathcal{O}_{C_p}^b$. Let $t = \log[\epsilon] \in B_{\text{dR}}^+$. Note that we also have $t \in A_{\text{cris}}$. Let $B_{\text{dR}} = B_{\text{dR}}^+[\frac{1}{t}]$, $B_{\text{cris}} = B_{\text{cris}}^+[\frac{1}{t}]$, and $B_{\text{st}} = B_{\text{st}}^+[\frac{1}{t}]$.

We denote by $\mathcal{O}_{\mathcal{E}}$ the p -adic completion of $\mathfrak{S}[\frac{1}{u}]$, and let $\mathcal{E} = \text{Frac}(\mathcal{O}_{\mathcal{E}})$. Let $\hat{\mathcal{E}}^{\text{ur}}$ be the p -adic completion of the maximal unramified subextension of \mathcal{E} in $W(\text{Frac}(\mathcal{O}_{C_p}^b))[\frac{1}{p}]$, and $\mathcal{O}_{\hat{\mathcal{E}}^{\text{ur}}}$ its ring of integers. We let $\mathfrak{S}^{\text{ur}} = \mathcal{O}_{\hat{\mathcal{E}}^{\text{ur}}} \cap W(\mathcal{O}_{C_p}^b)$.

We let $K'_{\infty} = \bigcup_{n=1}^{\infty} K'(\pi_n)$ and $K'_{p^{\infty}} = \bigcup_{n=1}^{\infty} K'(\zeta_{p^n})$. Let $K'_c = K'_{\infty} K'_{p^{\infty}}$, which is the Galois closure of K'_{∞} over K' . Let $\hat{\mathcal{G}} = \text{Gal}(K'_c/K')$, $\mathcal{G}_{\infty} = \text{Gal}(\bar{K}/K'_{\infty})$, and $\mathcal{H}_{K'} = \text{Gal}(K'_c/K'_{\infty})$. Write

$$t^{\{i\}} = \frac{t^i}{p^{q(i)} q(i)!}$$

where $q(i)$ is defined by $i = q(i)(p-1) + r(i)$ with $0 \leq r(i) < p-1$. We define

$$\mathcal{R}_{K_0} := \left\{ \sum_{i=0}^{\infty} a_i t^{\{i\}} \mid a_i \in S'_{K_0}, a_i \rightarrow 0 \text{ } p\text{-adically as } i \rightarrow \infty \right\}.$$

We have a natural map $\nu : W(\mathcal{O}_{C_p}^b) \rightarrow W(\bar{k})$ induced by the projection $\mathcal{O}_{C_p}^b \rightarrow \bar{k}$, which can be seen to extend uniquely to $\nu : B_{\text{cris}}^+ \rightarrow W(\bar{k})[\frac{1}{p}]$. For any subring $A \subset B_{\text{cris}}^+$, write $I_+ A := A \cap \ker(\nu)$. We have $I_+ \mathfrak{S} = u \mathfrak{S}$ and

$$I_+ S' = \left\{ \sum_{i=1}^{\infty} \frac{b_i}{[\frac{i}{e'}]!} u^i \mid b_i \in W(k), b_i \rightarrow 0 \text{ } p\text{-adically as } i \rightarrow \infty \right\}.$$

Define $\hat{\mathcal{R}} = W(\mathcal{O}_{C_p}^b) \cap \mathcal{R}_{K_0}$ and $I_+ = I_+ \hat{\mathcal{R}}$. The following lemma is proved in [Liu10].

Lemma 2.3. ([Liu10, Lemma 2.2.1])

1. $\hat{\mathcal{R}}$ (resp. \mathcal{R}_{K_0}) is a φ -stable \mathfrak{S} -algebra as a subring in $W(\mathcal{O}_{C_p}^b)$ (resp. B_{cris}^+).
2. $\hat{\mathcal{R}}$ and I_+ (resp. \mathcal{R}_{K_0} and $I_+ \mathcal{R}_{K_0}$) are $\mathcal{G}_{K'}$ -stable. The $\mathcal{G}_{K'}$ -actions on $\hat{\mathcal{R}}$ and I_+ (resp. \mathcal{R}_{K_0} and $I_+ \mathcal{R}_{K_0}$) factor through $\hat{\mathcal{G}}$.
3. $\mathcal{R}_{K_0}/I_+ \mathcal{R}_{K_0} \cong K_0$ and $\hat{\mathcal{R}}/I_+ \cong S'/I_+ S' \cong \mathfrak{S}/u \mathfrak{S} \cong W(k)$.

Let r be a positive integer. A *Kisin module of height r* is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ where \mathfrak{M} is a finite free \mathfrak{S} -module, and $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -semi-linear map such that the cokernel of the induced map $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ is killed by $F(u)^r$. A morphism between two Kisin modules $\mathfrak{M}_1, \mathfrak{M}_2$ is a morphism as \mathfrak{S} -modules compatible with $\varphi_{\mathfrak{M}_i}$. Let $\text{Mod}_{\mathfrak{S}}^r(\varphi)$ denote the category of Kisin modules of height r . For $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$, we write $\hat{\mathfrak{M}} = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. The Frobenius $\varphi_{\mathfrak{M}}$ on \mathfrak{M} naturally extends to $\hat{\mathfrak{M}}$ by $\varphi_{\hat{\mathfrak{M}}}(a \otimes m) = \varphi_{\hat{\mathcal{R}}}(a) \otimes \varphi_{\mathfrak{M}}(m)$.

Definition 2.4. A $(\varphi, \hat{\mathcal{G}})$ -module of height r is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{\mathcal{G}}_{\mathfrak{M}})$ satisfying the following:

- $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is Kisin module of height r .
- $\hat{\mathcal{G}}_{\mathfrak{M}}$ denotes a $\hat{\mathcal{R}}$ -semi-linear $\hat{\mathcal{G}}$ -action on $\hat{\mathfrak{M}}$ which commutes with $\varphi_{\mathfrak{M}}$ and induces a trivial action on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$.
- Considering \mathfrak{M} as a $\varphi(\mathfrak{S})$ -submodule of $\hat{\mathfrak{M}}$, we have $\mathfrak{M} \subset \hat{\mathfrak{M}}^{\mathcal{H}_{K'}}$.

A morphism between two $(\varphi, \hat{\mathcal{G}})$ -modules $\mathfrak{M}_1, \mathfrak{M}_2$ of height r is a morphism in $\text{Mod}_{\mathfrak{S}}^r(\varphi)$ which commutes with the $\hat{\mathcal{G}}$ -actions. We denote by $\text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$ the category of $(\varphi, \hat{\mathcal{G}})$ -modules of height r . For $\hat{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$, we associate a $\mathbf{Z}_p[\mathcal{G}_{K'}]$ -module $\hat{T}^{\vee}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(\mathcal{O}_{C_p}^b))$ with $\mathcal{G}_{K'}$ -action given by $g(f)(x) = g(f(g^{-1}(x)))$ for $g \in \mathcal{G}_{K'}$, $f \in \hat{T}^{\vee}(\hat{\mathfrak{M}})$. Here, $\mathcal{G}_{K'}$ -action on $\hat{\mathfrak{M}}$ is given by $\hat{\mathcal{G}}$ -action on $\hat{\mathfrak{M}}$. Moreover, for $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^r(\varphi)$, we associate a $\mathbf{Z}_p[\mathcal{G}_{\infty}]$ -module $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\text{ur}})$ similarly. The main result proved in [Liu10] is the following.

Theorem 2.5. (cf. [Liu10, Theorem 2.3.1, Proposition 3.1.3])

1. \hat{T}^{\vee} induces an anti-equivalence between $\text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$ and the category of $\mathcal{G}_{K'}$ -stable \mathbf{Z}_p -lattices in semi-stable representations of $\mathcal{G}_{K'}$ having Hodge-Tate weights in $[0, r]$.
2. \hat{T}^{\vee} induces a natural $W(\mathcal{O}_{C_p}^b)$ -linear injection

$$\hat{\iota} : W(\mathcal{O}_{C_p}^b) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} \rightarrow W(\mathcal{O}_{C_p}^b) \otimes_{\mathbf{Z}_p} \hat{T}^{\vee}(\hat{\mathfrak{M}})$$

such that $\hat{\iota}$ is compatible with Frobenius maps and $\mathcal{G}_{K'}$ -actions on both sides. Here, $\hat{T}^{\vee}(\hat{\mathfrak{M}}) := \text{Hom}_{\mathbf{Z}_p}(\hat{T}^{\vee}(\hat{\mathfrak{M}}), \mathbf{Z}_p)$.

3. There exists a natural isomorphism $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \xrightarrow{\cong} \hat{T}^{\vee}(\hat{\mathfrak{M}})$ of $\mathbf{Z}_p[\mathcal{G}_{\infty}]$ -modules.

To construct the functor M_{st} , we establish a connection between $(\varphi, \hat{\mathcal{G}})$ -modules and filtered (φ, N) -modules. Let $V \in \text{Rep}_{\mathbf{Q}_p}^{\text{pst}, K', r}$, and let $T \subset V$ be a \mathcal{G}_K -stable \mathbf{Z}_p -lattice. By Theorem 2.5, there exists a unique $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$ such that $\hat{T}^{\vee}(\hat{\mathfrak{M}}) = T$ as $\mathbf{Z}_p[\mathcal{G}_{K'}]$ -modules. Let $\mathscr{D} := S'_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ equipped with the Frobenius endomorphism given by $\varphi_{\mathscr{D}} = \varphi_{S'_{K_0}} \otimes \varphi_{\mathfrak{M}}$. Let $D = \mathscr{D}/(I_+ S'_{K_0})\mathscr{D}$, which is a finite K_0 -vector space equipped with the Frobenius induced from $\varphi_{\mathscr{D}}$. By [Bre97, Proposition 6.2.1.1], there exists a unique section $s : D \rightarrow \mathscr{D}$ compatible with the Frobenius morphisms on both sides. Thus, $\mathscr{D} = S'_{K_0} \otimes_{K_0} D$ if we identify D with $s(D)$. So $B_{\text{cris}}^+ \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} \cong B_{\text{cris}}^+ \otimes_{K_0} D$, and the map $\hat{\iota}$ given in Theorem 2.5 (2) induces a natural injection $D \hookrightarrow B_{\text{cris}}^+ \otimes_{\mathbf{Z}_p} T^{\vee}$ where $T^{\vee} := \text{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p)$.

On the other hand, the functor $D_{\text{st}}^{K'}$ induces an injection

$$\iota : B_{\text{st}}^+ \otimes_{K_0} D_{\text{st}}^{K'}(V) \rightarrow B_{\text{st}}^+ \otimes_{\mathbf{Q}_p} V^{\vee}$$

such that ι is compatible with Frobenius, monodromy, filtration, and $\mathcal{G}_{K'}$ -action on both sides. The following is proved in [Liu12].

Proposition 2.6. (cf. [Liu12, Proposition 2.6, Corollary 2.7, 2.8]) *There exists a unique K_0 -linear isomorphism $i : D_{\text{st}}^{K'}(V) \rightarrow D$ such that i is compatible with the Frobenius morphisms on both sides and makes the following diagram commutative:*

$$\begin{array}{ccc} D_{\text{st}}^{K'}(V) & \hookrightarrow & B_{\text{st}}^+ \otimes_{\mathbf{Z}_p} T^\vee \\ \downarrow i & & \downarrow \text{mod } u \\ D & \hookrightarrow & B_{\text{cris}}^+ \otimes_{\mathbf{Z}_p} T^\vee \end{array}$$

Furthermore, such i is functorial.

Note that

$$\mathfrak{M}/u\mathfrak{M} \cong \varphi^*(\mathfrak{M})/u\varphi^*(\mathfrak{M}) \subset \mathscr{D}/I_+ S'_{K_0} \mathscr{D} = D.$$

We set $M_{\text{st}}(T) \subset D_{\text{st}}^{K'}(V)$ to be the inverse image of $\varphi^*(\mathfrak{M})/u\varphi^*(\mathfrak{M})$ under the isomorphism $i : D_{\text{st}}^{K'}(V) \rightarrow D$ given in Proposition 2.6. $M_{\text{st}}(T)$ is a finite free $W(k)$ -lattice in $D_{\text{st}}^{K'}(V)$ stable under Frobenius. Furthermore, it is proved in [Liu12, Corollary 2.12, Proposition 2.15] that $M_{\text{st}}(T)$ is stable under \mathcal{G}_K -action and N on $D_{\text{st}}^{K'}(V)$. Thus, $M_{\text{st}}(T)$ is a lattice of the filtered (φ, N, Γ) -module $D_{\text{st}}^{K'}(V)$. And the association $M_{\text{st}}(\cdot)$ is a contravariant functor from $\text{Rep}_{\mathbf{Z}_p}^{\text{pst}, K', r}$ to $L^r(\varphi, N, \Gamma)$ since the isomorphism i in Proposition 2.6 is functorial.

2.3 Potentially Semi-stable Torsion Representations

We now associate torsion filtered (φ, N, Γ) -modules to potentially semi-stable torsion representations. Denote by $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ the category of torsion representations L semi-stable over K' and of height r , in a sense that there exist lattices $\mathcal{L}_1, \mathcal{L}_2 \in \text{Rep}_{\mathbf{Z}_p}^{\text{pst}, K', r}$ with a \mathcal{G}_K -equivariant injection $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ such that $L \cong \mathcal{L}_2/j(\mathcal{L}_1)$ as $\mathbf{Z}_p[\mathcal{G}_K]$ -modules, and L is killed by some power of p . Morphisms between two torsion representations in $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ are morphisms of $\mathbf{Z}_p[\mathcal{G}_K]$ -modules. We call such $(\mathcal{L}_1, \mathcal{L}_2, j)$ a *lift* of L . We will sometimes denote simply by j a lift of L . Note that a lift of $L \in \text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ is not unique. Let $L, L' \in \text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ with lifts $(\mathcal{L}_1, \mathcal{L}_2, j), (\mathcal{L}'_1, \mathcal{L}'_2, j')$ respectively. If $f : L \rightarrow L'$ is a morphism in $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$, we say a morphism $\tilde{f} : \mathcal{L}_2 \rightarrow \mathcal{L}'_2$ in $\text{Rep}_{\mathbf{Z}_p}^{\text{pst}, K', r}$ is a *lift* of f if $\tilde{f}(j(\mathcal{L}_1)) \subset j'(\mathcal{L}'_1)$ and \tilde{f} induces f .

We denote by $M_{\text{tor}}^{\text{fl}, r}(\varphi, N, \Gamma)$ the category whose objects are finite $W(k)$ -modules M killed by some power of p and endowed with the following structures:

- a Frobenius semilinear morphism $\varphi : M \rightarrow M$,
- $W(k)$ -linear map $N : M \rightarrow M$ satisfying $N\varphi = p\varphi N$,
- $W(k)$ -linear Γ -action on M which commutes with φ and N , and

- $M_{K'} := \mathcal{O}_{K' \otimes W(k)} M$ has decreasing filtration by $\mathcal{O}_{K'}$ -submodules such that $\text{Fil}^0 M_{K'} = M_{K'}$ and $\text{Fil}^{r+1} M_{K'} = 0$. Also, $\gamma(\text{Fil}^i M_{K'}) \subset \text{Fil}^i M_{K'}$ for any $\gamma \in \Gamma$.

Morphisms in $M_{\text{tor}}^{\text{fil},r}(\varphi, N, \Gamma)$ are $W(k)$ -linear maps compatible with above structures. For $L \in \text{Rep}_{\text{tor}}^{\text{pst},K',r}$ with a lift $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$, we can associate an object $M_{\text{st},j}(L) \in M_{\text{tor}}^{\text{fil},r}(\varphi, N, \Gamma)$ as follows. By Theorem 2.2, we have the morphism $M_{\text{st}}(j) : M_{\text{st}}(\mathcal{L}_2) \rightarrow M_{\text{st}}(\mathcal{L}_1)$ in $L^r(\varphi, N, \Gamma)$ corresponding to j , and $M_{\text{st}}(j)$ is injective by [Liu12, Corollary 3.8]. We set $M_{\text{st},j}(L) = M_{\text{st}}(\mathcal{L}_1)/M_{\text{st}}(j)(M_{\text{st}}(\mathcal{L}_2))$. Then $M_{\text{st},j}(L)$ has natural endomorphisms φ and N , and Γ -action induced from $M_{\text{st}}(\mathcal{L}_1)$. Furthermore, tensoring by $\mathcal{O}_{K'}$ on $M_{\text{st}}(j)$ gives the following exact sequence:

$$0 \rightarrow \mathcal{O}_{K'} \otimes_{W(k)} M_{\text{st}}(\mathcal{L}_2) \rightarrow \mathcal{O}_{K'} \otimes_{W(k)} M_{\text{st}}(\mathcal{L}_1) \xrightarrow{q} \mathcal{O}_{K'} \otimes_{W(k)} M_{\text{st},j}(L) \rightarrow 0.$$

We define the filtration on $M_{\text{st},j}(L)_{K'}$ by $\text{Fil}^i M_{\text{st},j}(L)_{K'} := q(\text{Fil}^i M_{\text{st}}(\mathcal{L}_1)_{K'})$. This gives $M_{\text{st},j}(L)$ a structure as an object in $M_{\text{tor}}^{\text{fil},r}(\varphi, N, \Gamma)$. By the snake lemma, we further have the following exact sequence of the associated graded modules:

$$0 \rightarrow \text{gr}^i(M_{\text{st}}(\mathcal{L}_2)_{K'}) \rightarrow \text{gr}^i(M_{\text{st}}(\mathcal{L}_1)_{K'}) \rightarrow \text{gr}^i(M_{\text{st},j}(L)_{K'}) \rightarrow 0.$$

If $f : L \rightarrow L'$ is a morphism in $\text{Rep}_{\text{tor}}^{\text{pst},K',r}$ with a lift $\tilde{f} : (\mathcal{L}_1, \mathcal{L}_2, j) \rightarrow (\mathcal{L}'_1, \mathcal{L}'_2, j')$, then it induces a morphism $M_{\text{st},\tilde{f}}(f) : M_{\text{st},j'}(L') \rightarrow M_{\text{st},j}(L)$ in $M_{\text{tor}}^{\text{fil},r}(\varphi, N, \Gamma)$.

Note that the above construction depends on the choice of the lift of L . However, the following theorem, which can be deduced directly from [Liu15] and [Liu12], shows that the construction depends on lifts only up to a constant.

Theorem 2.7. *There exists a constant c depending only on $F(u)$ and r such that the following statement holds: for any morphism $f : L \rightarrow L'$ in $\text{Rep}_{\text{tor}}^{\text{pst},K',r}$ with lifts j, j' of L, L' respectively, there exists a morphism $\tilde{h} : M_{\text{st},j'}(L') \rightarrow M_{\text{st},j}(L)$ in $M_{\text{tor}}^{\text{fil},r}(\varphi, N, \Gamma)$ such that*

- *if there exists a morphism of lifts $\tilde{f} : j \rightarrow j'$ which lifts f , then $\tilde{h} = p^c M_{\text{st},\tilde{f}}(f)$,*
- *let $f' : L' \rightarrow L''$ be a morphism in $\text{Rep}_{\text{tor}}^{\text{pst},K',r}$, j'' a lift of L'' , and $\tilde{h}' : M_{\text{st},j''}(L'') \rightarrow M_{\text{st},j'}(L')$ the morphism in $M_{\text{tor}}^{\text{fil},r}(\varphi, N, \Gamma)$ associated to f', j' , and j'' . If there exists a morphism of lifts $\tilde{g} : j \rightarrow j''$ which lifts $f' \circ f$, then $\tilde{h} \circ \tilde{h}' = p^{2c} M_{\text{st},\tilde{g}}(f' \circ f)$.*

Proof. It follows directly from [Liu12, Theorem 3.1] and [Liu15, Theorem 2.1.3, Remark 2.1.5]. \square

The following corollary is immediate.

Corollary 2.8. (cf. [Liu12, Corollary 3.2], [Liu15, Corollary 2.1.4]) *With notations as in Theorem 2.7, assume that $f : L \rightarrow L'$ is an isomorphism with the inverse $f^{-1} : L' \rightarrow L$. Let $\tilde{h}_1 : M_{\text{st},j}(L) \rightarrow M_{\text{st},j'}(L')$ be the morphism as in Theorem 2.7 associated to f^{-1}, j , and j' . Then $\tilde{h} \circ \tilde{h}_1 = p^{2c} \text{Id}$ on $M_{\text{st},j}(L)$ and $\tilde{h}_1 \circ \tilde{h} = p^{2c} \text{Id}$ on $M_{\text{st},j'}(L')$. Furthermore, the similar statement holds for the induced morphisms on $\text{gr}^i(M_{\text{st},j}(L)_{K'})$ and $\text{gr}^i(M_{\text{st},j'}(L')_{K'})$.*

2.4 Representation with Coefficient

Let A be a \mathbf{Z}_p -algebra, and denote by $\text{Rep}_A^{\text{pst}, K', r}$ the subcategory of $\text{Rep}_{\mathbf{Z}_p}^{\text{pst}, K', r}$ whose objects are A -modules such that \mathcal{G}_K -actions are A -linear. Morphisms in $\text{Rep}_A^{\text{pst}, K', r}$ are morphisms of $A[\mathcal{G}_K]$ -modules. Let $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ be the subcategory of $\text{Rep}_{\text{tor}}^{\text{pst}, K', r}$ whose objects have lifts in $\text{Rep}_A^{\text{pst}, K', r}$, and the morphisms in $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ are morphisms of $A[\mathcal{G}_K]$ -modules. For $L \in \text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$ having a lift $j : \mathcal{L}_1 \hookrightarrow \mathcal{L}_2$ in $\text{Rep}_A^{\text{pst}, K', r}$, note that $M_{\text{st}}(\mathcal{L}_1)$ and $M_{\text{st}}(\mathcal{L}_2)$ are naturally $A \otimes_{\mathbf{Z}_p} W(k)$ -modules, and thus so is $M_{\text{st}, j}(L)$.

Proposition 2.9. *Let $f : L \rightarrow L'$ be a morphism in $\text{Rep}_{\text{tor}, A}^{\text{pst}, K', r}$, and let j and j' be lifts in $\text{Rep}_A^{\text{pst}, K', r}$ of L and L' respectively. Then, the associated morphism $\tilde{h} : M_{\text{st}, j}(L) \rightarrow M_{\text{st}, j'}(L')$ in $M_{\text{tor}}^{\text{fil}, r}(\varphi, N, \Gamma)$ as in Theorem 2.7 is a morphism of $A \otimes_{\mathbf{Z}_p} W(k)$ -modules.*

Proof. It follows immediately from [Liu12, Proposition 3.13] and [Liu15, Lemma 4.2.4]. \square

3 Hodge-Tate Type and Galois Type

3.1 Hodge-Tate Type

Let E be a finite extension of \mathbf{Q}_p , and let B be a finite E -algebra. Let V_B be a free B -module of rank d equipped with a continuous \mathcal{G}_K -action. Suppose that as a representation of \mathcal{G}_K , V_B is semi-stable over K' , i.e., $V_B \in \text{Rep}_{\mathbf{Q}_p}^{\text{pst}, K'}$. Then V_B is de Rham over K , and we set $D_{\text{dR}}^K(V_B) = (B_{\text{dR}} \otimes_{\mathbf{Q}_p} V_B^\vee)^{\mathcal{G}_K}$. For any E -algebra A , we write $A_K := A \otimes_{\mathbf{Q}_p} K$.

Lemma 3.1. (cf. [Liu15, Lemma 4.1.2])

1. *Let B' be a finite B -algebra, and write $V_{B'} = B' \otimes_B V_B$. Then $D_{\text{dR}}^K(V_{B'}) \cong B' \otimes_B D_{\text{dR}}^K(V_B)$, and $\text{gr}^i(D_{\text{dR}}^{K'}(V_{B'})) \cong B' \otimes_B \text{gr}^i(D_{\text{dR}}^K(V_B))$.*
2. *$D_{\text{dR}}^K(V_B)$ is a free B_K -module of rank d .*

Proof. (1) is proved in [Liu15, Lemma 4.1.2]. For (2), since $D_{\text{dR}}^K(V_B) = K \otimes_{K_0} D_{\text{st}}^{K'}(V_B)$, it suffices to prove that $D_{\text{st}}^{K'}(V_B)$ is a free $B \otimes_{\mathbf{Q}_p} K_0$ -module of rank d . For any finite B -algebra B' , we can show similarly as in (1) that $D_{\text{st}}^{K'}(V_{B'}) \cong B' \otimes_B D_{\text{st}}^{K'}(V_B)$. Let $B_{\text{red}} = B/\mathcal{N}(B)$ where $\mathcal{N}(B)$ denotes the nilradical of B . B_{red} is a reduced Artinian ring, so there exists a ring isomorphism $B_{\text{red}} \cong \prod_{j=1}^m E_j$ for some field E_j finite over E . $E_j \otimes_{\mathbf{Q}_p} K_0$ is isomorphic to a finite direct product of fields, so $D_{\text{st}}^{K'}(V_{E_j}) \cong E_j \otimes_B D_{\text{st}}^{K'}(V_B)$ is finite projective as an $E_j \otimes_{\mathbf{Q}_p} K_0$ -module. Note that the Frobenius morphism on K_0 extends E_j -linearly to $E_j \otimes_{\mathbf{Q}_p} K_0$, and the extended Frobenius permutes the maximal ideals of $E_j \otimes_{\mathbf{Q}_p} K_0$ transitively. Therefore, $D_{\text{st}}^{K'}(V_{E_j})$ is a free $E_j \otimes_{\mathbf{Q}_p} K_0$ -module of rank d , and $D_{\text{st}}^{K'}(V_{B_{\text{red}}}) = B_{\text{red}} \otimes_B D_{\text{st}}^{K'}(V_B)$ is a free $B_{\text{red}} \otimes_{\mathbf{Q}_p} K_0$ -module of rank d .

Let $\{e_1, \dots, e_d\}$ be a $B_{\text{red}} \otimes_{\mathbf{Q}_p} K_0$ -basis of $D_{\text{st}}^{K'}(V_{B_{\text{red}}})$, and choose a lift $\hat{e}_i \in D_{\text{st}}^{K'}(V_B)$ of e_i . By Nakayama's lemma, $\{\hat{e}_1, \dots, \hat{e}_d\}$ generate $D_{\text{st}}^{K'}(V_B)$ as a $B \otimes_{\mathbf{Q}_p} K_0$ -module. Thus, we have a surjection of $B \otimes_{\mathbf{Q}_p} K_0$ -modules

$$f : \bigoplus_{i=1}^d (B \otimes_{\mathbf{Q}_p} K_0) \cdot \hat{e}_i \twoheadrightarrow D_{\text{st}}^{K'}(V_B).$$

As a K_0 -vector space, $\dim_{K_0} D_{\text{st}}^{K'}(V_B) = d \cdot \dim_{\mathbf{Q}_p} B$. Thus, f is an isomorphism, and $D_{\text{st}}^{K'}(V_B)$ is a free $B \otimes_{\mathbf{Q}_p} K_0$ -module of rank d . \square

Let D_E be a finite E -vector space such that $D_{E,K} := D_E \otimes_E K$ is equipped with a decreasing filtration $\text{Fil}^i D_{E,K}$ of $E \otimes_{\mathbf{Q}_p} K$ -modules and $\{i \mid \text{gr}^i D_{E,K} \neq 0\} \subset \{0, \dots, r\}$. We denote $\mathbf{v} = (D_{E,K}, \{\text{Fil}^i D_{E,K}\}_{i=0, \dots, r})$. We say that V_B has *Hodge-Tate type* \mathbf{v} if $\text{gr}^i D_{\text{dR}}^K(V_B) \cong B \otimes_E \text{gr}^i D_{E,K}$ as B_K -modules for all i .

Lemma 3.2. *For a finite B -algebra B' , $V_{B'}$ has Hodge-Tate type \mathbf{v} if V_B has Hodge-Tate type \mathbf{v} .*

Proof. It follows immediately from Lemma 3.1. \square

The goal of this subsection is to prove the following theorem:

Theorem 3.3. (cf. [Liu15, Theorem 4.3.4]) *There exists a constant c_1 depending only on K', r , and d such that the following statement holds:*

Let A and A' be finite flat \mathcal{O}_E -algebras and let $\rho : \mathcal{G}_K \rightarrow \text{GL}_d(A)$ and $\rho' : \mathcal{G}_K \rightarrow \text{GL}_d(A')$ be representations such that $\rho \in \text{Rep}_A^{\text{pst}, K', r}$ and $\rho' \in \text{Rep}_{A'}^{\text{pst}, K', r}$. Suppose that there exist an ideal $I \subset A$ such that A/I is killed by a power of p and an \mathcal{O}_E -linear surjection $\beta : A' \twoheadrightarrow A/I$ such that $A/I \otimes_A \rho \cong A/I \otimes_{\beta, A'} \rho'$ as $A[\mathcal{G}_K]$ -modules. Let V be the free $A[\frac{1}{p}]$ -module of rank d equipped with the \mathcal{G}_K -action corresponding to $\rho \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, and similarly let V' be corresponding to $\rho' \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. If $I \subset p^{c_1} A$ and V' has Hodge-Tate type \mathbf{v} , then V also has Hodge-Tate type \mathbf{v} .

When k is further assumed to be finite, Theorem 3.3 is proved in [Liu15, Theorem 4.3.4]. Some arguments of the proof in [Liu15] are based on reducing to the case when E contains the Galois closure of K' , and thus require k to be finite. We remove such a restriction in the following.

Since E is a finite extension of \mathbf{Q}_p , we have a ring isomorphism $E_K = E \otimes_{\mathbf{Q}_p} K \cong \prod_{j=1}^s H_j$ for some fields H_j finite over K . Note that each H_j is an E_K -algebra via $E_K \cong \prod_{i=1}^s H_i \xrightarrow{q_j} H_j$ where q_j is the natural projection onto the j -th factor. For any E_K -module M , we write $M_j := M \otimes_{E_K} H_j$. Then $M \cong \bigoplus_{j=1}^s M_j$. For a filtered E_K -module D_K , we denote $(\text{Fil}^i D_K)_j$ and $(\text{gr}^i D_K)_j$ by $\text{Fil}_j^i D_K$ and $\text{gr}_j^i D_K$ respectively. Since any finite E_K -module is projective, we have $\text{gr}_j^i D_K \cong \text{Fil}_j^i D_K / \text{Fil}_j^{i+1} D_K$. We write $B_{H_j} := B \otimes_E H_j$.

Lemma 3.4. (cf. [Liu15, Lemma 4.1.4]) *With notations as above, V_B has Hodge-Tate type \mathbf{v} if and only if $\mathrm{gr}_j^i D_{\mathrm{dR}}^K(V_B)$ is B_{H_j} -free and $\mathrm{rank}_{B_{H_j}} \mathrm{gr}_j^i D_{\mathrm{dR}}^K(V_B) = \dim_{H_j} \mathrm{gr}_j^i D_{E,K}$ for all $j = 1, \dots, s$ and $i \in \mathbf{Z}$.*

Proof. This follows from the same argument as in the proof of [Liu15, Lemma 4.1.4]. \square

$B_{\mathrm{red}} = N/\mathcal{N}(B)$ is a reduced Artinian E -algebra, so $B_{\mathrm{red}} \cong \prod_{l=1}^m E_l$ for some field E_l finite over E . We set $V_{E_l} = E_l \otimes_B V_B$.

Lemma 3.5. (cf. [Liu15, Proposition 4.1.5]) *V_B has Hodge-Tate type \mathbf{v} if and only if V_{E_l} has Hodge-Tate type \mathbf{v} for each $l = 1, \dots, m$.*

Proof. This follows from the same argument as in the proof of [Liu15, Proposition 4.1.5]. \square

The following lemma is useful when we consider an extension of the coefficient field E .

Lemma 3.6. *Let H be a field, and let C be a field (possibly of an infinite degree) over H . Let H' be a finite extension of H , and let R and T be finite extensions of H' . If M is a $C \otimes_H R$ -module such that $M \otimes_{H'} T$ is a finite free $C \otimes_H R \otimes_{H'} T$ -module, then M is finite free over $C \otimes_H R$.*

Proof. M is a finite projective $C \otimes_H R$ -module, and there exists a surjection $f : M \otimes_{H'} T \twoheadrightarrow M$ of $C \otimes_H R$ -modules having a section. Let $\{e_1, \dots, e_n\}$ be a basis of $M \otimes_{H'} T$ over $C \otimes_H R \otimes_{H'} T$. Let $N := \bigoplus_{i=1}^n (C \otimes_H R) \cdot f(e_i)$. Then the natural map $N \rightarrow M$ of $C \otimes_H R$ -modules is an injection since $\{e_1, \dots, e_n\}$ is a basis of $M \otimes_{H'} T$ over $C \otimes_H R \otimes_{H'} T$. Furthermore, $\dim_C N = \dim_C M$, so it is bijective. \square

Let L be a finite extension of E , and write $B_L := L \otimes_E B$. Given \mathbf{v} as above, let $\mathbf{v}' = (D_L := L \otimes_E D, \{\mathrm{Fil}^i D_{L,K} = L \otimes_E \mathrm{Fil}^i D_{E,K}\}_{i=0, \dots, r})$.

Lemma 3.7. (cf. [Liu15, Lemma 4.1.6]) *With notations as above, V_B has Hodge-Tate type \mathbf{v} if and only if $V_{B_L} := B_L \otimes_B V_B$ has Hodge-Tate type \mathbf{v}' .*

Proof. Given Lemma 3.6, it follows from the same argument as in the proof of [Liu15, Lemma 4.1.6]. \square

Lemma 3.8. *Suppose we have an injection $B \hookrightarrow B'$ of finite E -algebras. If $V_{B'} = B' \otimes_B V_B$ has Hodge-Tate type \mathbf{v} , then also V_B has Hodge-Tate type \mathbf{v} .*

Proof. We have an induced injection of finite E -algebras $B_{\mathrm{red}} \hookrightarrow B'_{\mathrm{red}}$. By Lemma 3.5, we can reduce to the case when B and B' are fields. Then it follows from Lemma 3.4 and Lemma 3.6. \square

As we will apply the functor M_{st} to \mathcal{G}_K -representations semi-stable over K' , we need to consider $D_{\text{dR}}^{K'}(V_B) := (B_{\text{dR}} \otimes_{\mathbf{Q}_p} V_B^\vee)^{\mathcal{G}_{K'}}$. Note that $D_{\text{dR}}^{K'}(V_B) = D_{\text{dR}}^K(V_B) \otimes_K K'$. Thus, by essentially the same argument as in the proof of Lemma 3.7, we see that V_B has Hodge-Tate type \mathbf{v} if and only if $\text{gr}^i D_{\text{dR}}^{K'}(V_B) \cong B \otimes_E \text{gr}^i D_{E,K'}$ as $B_{K'}$ -modules for all i . Here, $D_{E,K'} := D_E \otimes_E K' = D_{E,K} \otimes_K K'$ which has the induced filtration from $D_{E,K}$.

Let $K_1 \subset E$ be the maximal unramified subextension over \mathbf{Q}_p . Then $K_1 = W(k_1)[\frac{1}{p}]$ for some finite field k_1 , and E/K_1 is totally ramified. Choose a uniformizer $\varpi_E \in E$, and let $\tilde{F}(u)$ be its Eisenstein polynomial over K_1 . Let $G(u)$ be a monic irreducible polynomial in $\mathbf{Q}_p[u]$ such that $K_1 \cong \mathbf{Q}_p[u]/G(u)\mathbf{Q}_p[u]$, and let $G(u) = \prod_{j=1}^m G_j(u)$ be the decomposition into monic irreducible polynomials in $K_0[u]$. Note that $G_j(u) \in W(k)[u]$. Denote by $\bar{G}_j(u) \in k[u]$ the reduction of $G_j(u)$ mod p . Then $\bar{G}_j(u)$ is irreducible in $k[u]$ and $k[u]/\bar{G}_j(u)k[u] \cong l_j$ for a finite extension l_j/k . By the Chinese remainder theorem, $W(k_1) \otimes_{\mathbf{Z}_p} W(k) \cong \prod_{j=1}^m W(l_j)$. Since $\tilde{F}(u)$ is irreducible over $W(l_j)[\frac{1}{p}]$ for each j , we have $E \otimes_{\mathbf{Q}_p} K_0 \cong \prod_{j=1}^m L_j$ and $\mathcal{O}_E \otimes_{\mathbf{Z}_p} W(k) \cong \prod_{j=1}^m \mathcal{O}_{L_j}$ where $L_j := (W(l_j)[\frac{1}{p}])(\varpi_E)$.

For each $j = 1, \dots, m$, let $F(u) = \prod_{s=1}^{t_j} F_{j_s}(u)$ be the decomposition of $F(u)$ into monic irreducible polynomials in $L_j[u]$, and choose a root ϖ_{j_s} of $F_{j_s}(u)$ for each s . Then

$$L_j \otimes_{K_0} K' \cong \prod_{s=1}^{t_j} L_j[u]/F_{j_s}(u)L_j[u] \cong \prod_{s=1}^{t_j} T_{j_s}$$

where $T_{j_s} := L_j(\varpi_{j_s})$. Thus, we have ring isomorphisms

$$E \otimes_{\mathbf{Q}_p} K' \cong (E \otimes_{\mathbf{Q}_p} K_0) \otimes_{K_0} K' \cong \prod_{j=1}^m \prod_{s=1}^{t_j} T_{j_s}.$$

Let $t = \sum_{j=1}^m t_j$. Then after re-indexing the fields T_{j_s} , we have $E \otimes_{\mathbf{Q}_p} K' \cong \prod_{s=1}^t T_j$, and the statement analogous to Lemma 3.4 holds for $D_{\text{dR}}^{K'}(V_B)$.

Let $\mathcal{O}_{E,K'} := \mathcal{O}_E \otimes_{\mathbf{Z}_p} \mathcal{O}_{K'}$. The projection $q_s : E_{K'} \rightarrow T_s$ induces the map $\mathcal{O}_{E,K'} \rightarrow \mathcal{O}_{T_s}$, and we have the natural map $q : \mathcal{O}_{E,K'} \rightarrow \prod_{s=1}^t \mathcal{O}_{T_s}$. Denote by v_p the p -adic valuation normalized by $v_p(p) = 1$.

Lemma 3.9. *There exists a positive integer c' depending only on K_0 and $F(u)$ such that $p^{c'}(\prod_{s=1}^t \mathcal{O}_{T_s}) \subset q(\mathcal{O}_{E,K'})$.*

Proof. For a field L finite over K_0 , let $F(u) = \prod_{s=1}^w F_s(u)$ be the decomposition of $F(u)$ into monic irreducible polynomials in $L[u]$, and choose a root ϖ_s of $F_s(u)$ for each s . We have

$$L \otimes_{K_0} K' \cong \prod_{s=1}^w L[u]/F_s(u)L[u] \cong \prod_{s=1}^w L'_s$$

where $L'_s := L(\varpi_s)$. Let $q'_s : L \otimes_{K_0} K' \rightarrow L'_s$ be the composition of the above isomorphism followed by the projection onto the s -th factor. Then q'_s induces a surjection $\mathcal{O}_L \otimes_{\mathbf{Z}_p}$

$\mathcal{O}_{K'} \twoheadrightarrow \mathcal{O}_{L'_s}$, and we have the natural map $\mathcal{O}_L \otimes_{\mathbf{Z}_p} \mathcal{O}_{K'} \hookrightarrow \prod_{s=1}^w \mathcal{O}_{L'_s}$. Under this map, $\prod_{h \neq s} F_h(\pi)$ maps to $(0, \dots, 0, \prod_{h \neq s} F_h(\varpi_s), 0, \dots, 0)$ whose components are 0 except the s -th component. Write $v_p(\prod_{h \neq s} F_h(\varpi_s)) = \frac{a}{b}$ for some relatively prime positive integers a, b . Then $(\prod_{h \neq s} F_h(\varpi_s))^b = p^a x$ for some $x \in \mathcal{O}_{L'_s}^\times$ with $v_p(x) = 0$. Thus, $(0, \dots, 0, p^a, 0, \dots, 0)$, whose components are 0 except the s -th component, lies in the image of $\mathcal{O}_L \otimes_{\mathbf{Z}_p} \mathcal{O}_{K'}$ under the above map.

Repeating this argument for all $s = 1, \dots, w$ and considering all possible decompositions of $F(u)$ into irreducible factors over some finite field over K_0 , we see that there exists a positive integer c' depending only on K_0 and $F(u)$ such that for any L finite over K_0 , if we write $L \otimes_{K_0} K \cong \prod_{s=1}^w L(\varpi_s)$ as above, then for each s , $(0, \dots, 0, p^{c'}, 0, \dots, 0)$ whose components are 0 except the s -th component lies in the image of $\mathcal{O}_L \otimes_{\mathbf{Z}_p} \mathcal{O}_{K'}$. Applying this for each L_j , we get the result. \square

Corollary 3.10. *Let M be a torsion free $\mathcal{O}_{E,K'}$ -module. Then the torsion part of $M_s := M \otimes_{\mathcal{O}_{E,K'}, q_s} \mathcal{O}_{T_s}$ is killed by $p^{c'}$, where c' is the constant given in Lemma 3.9.*

Proof. Let $M' = \bigoplus_{s=1}^t M_s$. By Lemma 3.9, there exist morphisms of $\mathcal{O}_{E,K'}$ -modules $q_M : M \rightarrow M'$ and $s_M : M' \rightarrow M$ such that $q_M \circ s_M = p^{c'} \text{Id}|_{M'}$. Let x be a torsion element in M' . Then $s_M(x) = 0$, so $p^{c'} x = q_M(s_M(x)) = 0$. \square

Let C be a finite flat \mathcal{O}_E -algebra, and let $\Lambda \in \text{Rep}_C^{\text{pst}, K', r}$ such that Λ is a finite free C -module of rank d and $\Lambda[\frac{1}{p}]$ has Hodge-Tate type \mathbf{v} . Since \mathcal{O}_E is henselian, $C \cong \prod_{j=1}^n C_j$ where each C_j is a finite flat local \mathcal{O}_E -algebra. We say C is *good* if for each $j = 1, \dots, n$, there exists a prime ideal $\mathfrak{p}_j \subset C_j$ such that $C_j/\mathfrak{p}_j \cong \mathcal{O}_{F_j}$ for some finite extension F_j/\mathbf{Q}_p . Let $L_{K'} := M_{\text{st}}(\Lambda)_{K'}$.

Lemma 3.11. *Suppose C is good. Then $L_{K'}$ is finite free over $C_{K'} := C \otimes_{\mathbf{Z}_p} \mathcal{O}_{K'}$ of rank d .*

Proof. By Theorem 2.5, there exists a unique Kisin module $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^r(\varphi, \hat{\mathcal{G}})$ such that $\hat{T}^\vee(\mathfrak{M}) = \Lambda$. Write $\mathfrak{S}_C := C \otimes_{\mathbf{Z}_p} \mathfrak{S}$. From the construction of the functor M_{st} in Section 2.2, it suffices to show that \mathfrak{M} is a finite free \mathfrak{S}_C -module of rank d . The Kisin module corresponding to $C_j \otimes_C \Lambda$ is $C_j \otimes_C \mathfrak{M}$, so we may assume without loss of generality that $n = 1$ and that C is a local ring.

The Kisin module corresponding to $\mathcal{O}_{F_1} \otimes_C \Lambda$ (via $C/\mathfrak{p}_1 \cong \mathcal{O}_{F_1}$) is $\mathfrak{M}' := \mathcal{O}_{F_1} \otimes_C \mathfrak{M}$. Since \mathfrak{M}' is finite free over \mathfrak{S} , $\mathfrak{M}'/u\mathfrak{M}'$ is p -torsion free. Thus, $\mathfrak{M}'/u\mathfrak{M}'$ is a projective $\mathcal{O}_{F_1} \otimes_{\mathbf{Z}_p} W(k)$ -module. Since $(\mathfrak{M}'/u\mathfrak{M}')[\frac{1}{p}]$ is isomorphic to its pullback by φ and φ permutes the maximal ideals of $\mathcal{O}_{F_1} \otimes_{\mathbf{Z}_p} W(k)$ transitively, $\mathfrak{M}'/u\mathfrak{M}'$ is a free $\mathcal{O}_{F_1} \otimes_{\mathbf{Z}_p} W(k)$ -module of rank d . Thus, \mathfrak{M}' is a free $\mathcal{O}_{F_1} \otimes_{\mathbf{Z}_p} \mathfrak{S}$ -module of rank d .

By Nakayama's lemma, we have a surjection

$$f : \bigoplus_{i=1}^d \mathfrak{S}_C \cdot e_i \twoheadrightarrow \mathfrak{M}$$

of \mathfrak{S}_C -modules. Λ is a free \mathbf{Z}_p -module of rank $[C : \mathbf{Z}_p]d$, so \mathfrak{M} is free over \mathfrak{S} of rank $[C : \mathbf{Z}_p]d$. Thus, f is an isomorphism. \square

Suppose that C is good and that there exists an ideal $J \subset C$ such that $C/J \cong \mathcal{O}_E/p^b\mathcal{O}_E$ for some positive integer b . For $s = 1, \dots, t$, we set $C[\frac{1}{p}]_s := (C[\frac{1}{p}] \otimes_{\mathbf{Q}_p} K') \otimes_{E_{K'}, q_s} T_s$, and define $d_s := \text{rank}_{C[\frac{1}{p}]_s}(\text{gr}_s^0(D_{\text{dR}}^{K'}(\Lambda[\frac{1}{p}])))$. Denote $\text{Fil}_s^i L_{K'} := \text{Fil}^i L_{K'} \otimes_{\mathcal{O}_{E, K'}, q_s} \mathcal{O}_{T_s}$, and similarly for the graded modules. By Lemma 3.11, $\text{Fil}_s^0 L_{K'}$ is free over $C_s := C_{K'} \otimes_{\mathcal{O}_{E, K'}, q_s} \mathcal{O}_{T_s}$ of rank d .

Lemma 3.12. (cf. [Liu15, Lemma 4.2.7]) *Suppose that $d_s \neq 0$. Let l be a positive integer satisfying $b \geq ld + 1$. Then there exists $x \in \text{gr}_s^0 L_{K'}/J \text{gr}_s^0 L_{K'}$ such that $p^l x \neq 0$.*

Proof. This follows from essentially the same argument as in the proof of [Liu15, Lemma 4.2.7]. For any C -module M , denote M/JM by M/J . We have the following right exact sequence:

$$\text{Fil}_s^1 L_{K'} \rightarrow \text{Fil}_s^0 L_{K'} \rightarrow \text{gr}_s^0 L_{K'} \rightarrow 0.$$

Let $\tilde{\text{Fil}}_s^1 L_{K'}$ be the image of $\text{Fil}_s^1 L_{K'}$ in $\text{Fil}_s^0 L_{K'}$ under the first map in the above sequence. We then obtain the following right exact sequence

$$\tilde{\text{Fil}}_s^1 L_{K'}/J \rightarrow \text{Fil}_s^0 L_{K'}/J \rightarrow \text{gr}_s^0 L_{K'}/J \rightarrow 0.$$

Denote $\bar{M} := \text{Fil}_s^0 L_{K'}/J$ and let $\bar{N} \subset \bar{M}$ be the submodule given by the image of $\tilde{\text{Fil}}_s^1 L_{K'}/J$. Then $\bar{M}/\bar{N} = \text{gr}_s^0 L_{K'}/J$.

Suppose that p^l annihilates \bar{M}/\bar{N} . By Lemma 3.11, \bar{M} is a finite free $\mathcal{O}_{T_s}/p^b\mathcal{O}_{T_s}$ -module of rank d . Let $\bar{\pi}_s$ be a uniformizer of \mathcal{O}_{T_s} . Then there exists an $\mathcal{O}_{T_s}/p^b\mathcal{O}_{T_s}$ -basis $\bar{e}_1, \dots, \bar{e}_d$ of \bar{M} such that

$$\bar{N} \cong \bigoplus_{i=1}^d (\mathcal{O}_{T_s}/p^b\mathcal{O}_{T_s}) \cdot (\bar{\pi}_s^{a_i} \bar{e}_i)$$

for some nonnegative integers a_i . We have $\bar{\pi}_s^{a_i} \mid p^l$ for all $i = 1, \dots, d$. Let e_1, \dots, e_d be a C_s -basis of $\text{Fil}_s^0 L_{K'}$ which lifts $\bar{e}_1, \dots, \bar{e}_d$. For $i = 1, \dots, d$, let $y_i \in \tilde{\text{Fil}}_s^1 L_{K'}$ which lifts $\bar{\pi}_s^{a_i} \bar{e}_i$. If X denotes the $d \times d$ -matrix such that $(y_1, \dots, y_d) = (e_1, \dots, e_d)X$, then $\det(X) = \bar{\pi}_s^a + j$ with $a = \sum_{i=1}^d a_i$ and $j \in J$. Since $b \geq ld + 1$, we have $\bar{\pi}_s^a \neq 0$ in C_s/J , and thus $\det(X) \neq 0$ in C_s . On the other hand, let $\bar{z}_1, \dots, \bar{z}_d$ be a $C[\frac{1}{p}]_s$ -basis of $\text{gr}_s^0(D_{\text{dR}}^{K'}(\Lambda[\frac{1}{p}])))$. We have $\det(X)(e_1, \dots, e_d) \subset \text{Fil}_s^1(D_{\text{dR}}^{K'}(\Lambda[\frac{1}{p}])))$, and therefore $\det(X)\bar{z}_i = 0$. This gives a contradiction. \square

Proof of Theorem 3.3. Given above results, Theorem 3.3 follows from essentially the same argument as in the proof of [Liu15, Theorem 4.3.4], except that we do not reduce to the case where E contains the Galois closure of K' . We also remark that the proof of [Liu15]

reduces to the case A' is local. But A' is not necessarily finite over \mathcal{O}_E after the reduction, which has been overlooked in [Liu15]. This is a very minor gap, and we remedy it by only reducing to the case A' is good.

We first reduce to the case where $A = \mathcal{O}_E$ and A' is good. For this, let $B := A \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $B' := A' \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. We have $B_{\text{red}} = B/\mathcal{N}(B) \cong \prod_j E_j$ and $B'_{\text{red}} \cong \prod E'_j$ for some E_j, E'_j finite over E . Let L be a finite Galois extension of E containing all Galois closures of E_j, E'_j . Denote $\mathcal{O}_L \otimes_{\mathcal{O}_E} (*)$ by $(*)_{\mathcal{O}_L}$ for $(*)$ being A, A', ρ, ρ', I , and β . Note that $(A_{\mathcal{O}_L}[\frac{1}{p}])_{\text{red}} = L \otimes_E B_{\text{red}} = L \otimes_E \prod E_j \cong \prod_i L$ with E_j embedding into L differently, and similarly for $(A'_{\mathcal{O}_L}[\frac{1}{p}])_{\text{red}}$. This induces the natural map $\psi_l : A_{\mathcal{O}_L} \rightarrow (A_{\mathcal{O}_L}[\frac{1}{p}])_{\text{red}} \rightarrow L$ to the l -th factor of $\prod_i L$. By Lemma 3.5 and 3.7, it suffices to show (assuming $I \subset p^{c_1}A$ for a suitable constant c_1) that $L \otimes_{\psi_l, A_{\mathcal{O}_L}} (\rho)_{\mathcal{O}_L}$ has Hodge-Tate type \mathbf{v} . Let $A_l = \psi_l(A_{\mathcal{O}_L})$ and $I_l = \psi_l(I_{\mathcal{O}_L})$. $\psi_l : A_{\mathcal{O}_L} \twoheadrightarrow A_l \subset L$ is a morphism of \mathcal{O}_L -algebras, so $A_l = \mathcal{O}_L$ (and analogously for $A'_{\mathcal{O}_L}$), and we have a natural projection $\gamma_l : A_{\mathcal{O}_L}/I_{\mathcal{O}_L} \twoheadrightarrow A_l/I_l$. Thus, by replacing E by L and A' by $A'_{\mathcal{O}_L}$, we can assume that $A = \mathcal{O}_E$ and that A' is good.

Let T denote the torsion representation $A/I \otimes_A \rho \cong A'/I' \otimes_{A'} \rho' \in \text{Rep}_{\text{tor}, \mathcal{O}_E}^{\text{pst}, K', r}$ where $I' = \ker(\beta)$. We denote by j and j' the two lifts ρ and ρ' of T respectively. Write $L_{K'} := M_{\text{st}}(\rho)_{K'}, L'_{K'} := M_{\text{st}}(\rho')_{K'}, M_{K'} := M_{\text{st}, j}(T)_{K'}$, and $M'_{K'} := M_{\text{st}, j'}(T)_{K'}$. We have $\text{gr}_s^i M_{K'} \cong \text{gr}_s^i L_{K'}/I \text{gr}_s^i L_{K'}$ and $\text{gr}_s^i M'_{K'} \cong \text{gr}_s^i L'_{K'}/I' \text{gr}_s^i L'_{K'}$ for $s = 1, \dots, t$. By Corollary 2.8 and Proposition 2.9, there exist morphisms of \mathcal{O}_{T_s} -modules $g_s^i : \text{gr}_s^i M_{K'} \rightarrow \text{gr}_s^i M'_{K'}$ and $h_s^i : \text{gr}_s^i M'_{K'} \rightarrow \text{gr}_s^i M_{K'}$ such that $g_s^i \circ h_s^i = p^{2c} \text{Id}|_{\text{gr}_s^i M'_{K'}}$ and $h_s^i \circ g_s^i = p^{2c} \text{Id}|_{\text{gr}_s^i M_{K'}}$.

Now, we set $\tilde{c} = \tilde{c}(K', r, d) := (2c + c')d + 1$ where c and c' are given as in Theorem 2.7 and Lemma 3.9 respectively. Assume $I \subset p^{\tilde{c}}A = p^{\tilde{c}}\mathcal{O}_E$. We claim that if $\text{gr}_s^0(D_{\text{dR}}^{K'}(V')) \neq 0$, then $\text{gr}_s^0(D_{\text{dR}}^{K'}(V)) \neq 0$. Suppose $\text{gr}_s^0(D_{\text{dR}}^{K'}(V)) = 0$. By Corollary 3.10, $\text{gr}_s^0 M_{K'}$ is killed by $p^{c'}$. But by Lemma 3.12, there exists $x \in \text{gr}_s^0 M'_{K'}$ such that $p^{c'+2c}x \neq 0$. We have a contradiction since $p^{c'+2c}x = g_s^0(p^{c'}h_s^0(x))$.

On the other hand, let $B' = A'[\frac{1}{p}]$, and denote $d_0 = \dim_{T_s} \text{gr}_s^0(D_{\text{dR}}^{K'}(V))$. We claim (assuming $I \subset p^{\tilde{c}}\mathcal{O}_E$) that $d_0 \leq \dim_{T_s} \text{gr}_s^0(D_{\text{dR}}^{K'}(V'))$. For this, note that as an \mathcal{O}_{T_s} -module, $\text{gr}_s^0 L_{K'} = N_{\text{tor}} \oplus N$ where N_{tor} is the torsion submodule of $\text{gr}_s^0 L_{K'}$ and N is a finite free \mathcal{O}_{T_s} -module of rank d_0 . By Corollary 3.10,

$$\text{gr}_s^0 M_{K'} \cong N_{\text{tor}} \oplus \bigoplus_{i=1}^{d_0} \mathcal{O}_{T_s}/I\mathcal{O}_{T_s}.$$

Let $\bar{N} := p^{c'} \bigoplus_{i=1}^{d_0} \mathcal{O}_{T_s}/I\mathcal{O}_{T_s}$. Then $p^{c'} \text{gr}_s^0 M_{K'} = \bar{N}$, again by Corollary 3.10, and therefore $h_s^0(g_s^0(\bar{N})) \cong \bigoplus_{i=1}^{d_0} p^{2c+c'} \mathcal{O}_{T_s}/I\mathcal{O}_{T_s}$. Since $p^{c'} \text{gr}_s^0 L'_{K'}$ surjects onto $h_s^0(p^{c'} \text{gr}_s^0 M'_{K'})$ and $g_s^0(\bar{N}) \subset p^{c'} \text{gr}_s^0 M'_{K'}$, we have by Corollary 3.10 that the \mathcal{O}_{T_s} -rank of $p^{c'} \text{gr}_s^0 L'_{K'}$ is at least d_0 . Thus, the \mathcal{O}_{T_s} -rank of $\text{gr}_s^0 L'_{K'}$ is at least d_0 , and $\dim_{T_s} \text{gr}_s^0(D_{\text{dR}}^{K'}(V')) \geq d_0$.

Hence, assuming $I \subset p^{\tilde{c}}\mathcal{O}_E$, we have $\text{gr}_s^0(D_{\text{dR}}^{K'}(V)) \neq 0$ if and only if $\text{gr}_s^0(D_{\text{dR}}^{K'}(V')) \neq 0$.

For the last step, we set $c_1 = \tilde{c}(K', dr, d)$ and assume $I \subset p^{c_1}\mathcal{O}_E$. It suffices to show

that for each i ,

$$\dim_{T_s} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V)) = \mathrm{rank}_{B'_{T_s}} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V')).$$

Suppose that the above equality fails for some i , and let i_* be the smallest such number. Write $d_i = \dim_{T_s} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V))$ and $d'_i = \mathrm{rank}_{B'_{T_s}} \mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(V'))$. Suppose first $d_{i_*} > d'_{i_*}$. We set $t_1 = \sum_{i \leq i_*} d_i$ and $t_2 = \sum_{i \leq i_*} i d_i$. Let $\tilde{i} = \max\{i \mid \sum_{j \leq i} d'_j \leq t_1\}$ and $t' = \sum_{i \leq \tilde{i}} d'_i$. Then $i_* \leq \tilde{i}$ and $t' \leq t_1$. Let

$$t'' = \left(\sum_{i \leq \tilde{i}} i d'_i \right) + (t_1 - t')(\tilde{i} + 1).$$

We have $t_2 < t''$. Moreover, t_2 (resp. t'') is the smallest i such that $\mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(\bigwedge^{t_1} V))$ (resp. $\mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(\bigwedge^{t_1} V'))$) is nontrivial. Let χ be a crystalline character such that $\mathrm{gr}_s^i(D_{\mathrm{dR}}^{K'}(\chi)) \neq 0$ only when $i = -t_2$. Then $\mathrm{gr}_s^0(D_{\mathrm{dR}}^{K'}(\chi \bigwedge^{t_1} V))$ is nontrivial. From the above result applied to $\chi \bigwedge^{t_1} V$ and $\chi \bigwedge^{t_1} V'$, we see that $\mathrm{gr}_s^0(D_{\mathrm{dR}}^{K'}(\chi \bigwedge^{t_1} V'))$ is also nontrivial, leading to a contradiction.

By switching the roles of V and V' , it follows similarly that we cannot have $d_{i_*} < d'_{i_*}$. This completes the proof. \square

3.2 Galois Type

We now study the Galois types of potentially semi-stable representations. As in Section 3.1, let E be a finite field over \mathbf{Q}_p , and let B be a finite E -algebra. Let V_B be a free B -module of rank d equipped with a potentially semi-stable continuous \mathcal{G}_K -action. Let

$$D_{\mathrm{pst}}(V_B) = \lim_{K \subset \bar{K}''} (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V_B^\vee)^{\mathcal{G}_{K''}},$$

where the limit goes over finite extensions K'' of K contained in \bar{K} . Denote by K_0^{ur} the union of finite unramified extensions of K_0 contained in \bar{K} . We have $\dim_{K_0^{\mathrm{ur}}} D_{\mathrm{pst}}(V_B) = \dim_{\mathbf{Q}_p} V_B$.

Lemma 3.13. *Let B' be a finite B -algebra, and write $V_{B'} = B' \otimes_B V_B$. Then $V_{B'}$ is potentially semi-stable as a \mathcal{G}_K -representation, and $D_{\mathrm{pst}}(V_{B'}) \cong B' \otimes_B D_{\mathrm{pst}}(V_B)$. If V_B becomes semi-stable over $L \supset K$, then so does $V_{B'}$. Furthermore, $D_{\mathrm{pst}}(V_B)$ is a free $B \otimes_{\mathbf{Q}_p} K_0^{\mathrm{ur}}$ -module of rank d .*

Proof. It follows from essentially the same proof as for Lemma 3.1. \square

$D_{\mathrm{pst}}(V_B)$ is equipped with a K_0^{ur} -semilinear action of \mathcal{G}_K , and thus a K_0^{ur} -linear action of the inertia group I_K . The Frobenius action commutes with the I_K -action, so $\mathrm{tr}(\sigma | D_{\mathrm{pst}}(V_B)) \in B$ for all $\sigma \in I_K$.

Let D_E be an E -vector space of dimension d , and let $D_{E,K} = D_E \otimes_{\mathbf{Q}_p} K$ equipped with a filtration giving a Hodge-Tate type \mathbf{v} . Fix a representation

$$\tau : I_K \rightarrow \mathrm{End}_E(D_E)$$

with an open kernel. Note that there exists an I_K -stable \mathcal{O}_E -lattice in D_E , so $\mathrm{tr}(\tau(\sigma)) \in \mathcal{O}_E$ for all $\sigma \in I_K$. We say V_B has *Galois type* τ if the I_K -representation $D_{\mathrm{pst}}(V_B)$ is equivalent to τ , i.e., $\mathrm{tr}(\sigma|D_{\mathrm{pst}}(V_B)) = \mathrm{tr}(\tau(\sigma))$ for all $\sigma \in I_K$.

Let L/K be a finite Galois extension contained in \bar{K} such that $I_L \subset \ker(\tau)$. Here, I_L denotes the inertia subgroup of \mathcal{G}_L . $D_{\mathrm{st}}^L(V_B) = (B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V_B^\vee)^{\mathcal{G}_L}$ is an L_0 -vector space where L_0 is the maximal unramified subextension of K_0 contained in L . If V_B is semi-stable over L , then $D_{\mathrm{pst}}(V_B) \cong K_0^{\mathrm{ur}} \otimes_{L_0} D_{\mathrm{st}}^L(V_B)$. Therefore, V_B has Galois type τ if and only if V_B is semi-stable over L and $\mathrm{tr}(\sigma|D_{\mathrm{st}}^L(V_B)) = \mathrm{tr}(\tau(\sigma))$ for all $\sigma \in I_{L/K}$, where $I_{L/K}$ is the inertia subgroup of $\mathrm{Gal}(L/K)$.

Lemma 3.14. *Let $\alpha : B \rightarrow B'$ be an E -algebra morphism between finite E -algebras. Suppose V is semi-stable over L . Then for all $\sigma \in I_{L/K}$, we have $\mathrm{tr}(\sigma|D_{\mathrm{st}}^L(V_{B'})) = \alpha(\mathrm{tr}(\sigma|D_{\mathrm{st}}^L(V_B)))$. In particular, if V_B has Galois type τ , then so does $V_{B'}$. If α is injective, then the converse is also true, i.e., V_B has Galois type τ if and only if $V_{B'}$ has Galois type τ .*

Proof. $D_{\mathrm{pst}}(V_{B'}) \cong B' \otimes_B D_{\mathrm{pst}}(V_B)$ by Lemma 3.13, so

$$\mathrm{tr}(\sigma|D_{\mathrm{st}}^L(V_{B'})) = \alpha(\mathrm{tr}(\sigma|D_{\mathrm{st}}^L(V_B)))$$

for all $\sigma \in I_{L/K}$. The remaining statements follow immediately. \square

Consider the case when B is local. If E' is its residue field, then E' is finite over E and B is naturally an E' -algebra. Note that the I_K -action on $D_{\mathrm{pst}}(V_B)$ has an open kernel. Since the cohomology of a finite group with coefficients in $E' \otimes_{\mathbf{Q}_p} K_0^{\mathrm{ur}}$ is trivial in all positive degrees, it follows from the deformation theory that the representation $D_{\mathrm{pst}}(V_B)$ arises from a representation over $E' \otimes_{\mathbf{Q}_p} K_0^{\mathrm{ur}}$. Thus, V_B has Galois type τ if and only if $V_{E'} = E' \otimes_B V_B$ has Galois type τ . For a general finite E -algebra B , we have isomorphisms $B \cong \prod_{i=1}^n B_{\mathfrak{m}_i}$ and $B_{\mathrm{red}} \cong \prod_{i=1}^n E_i$, where $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are the maximal ideals of B and $E_i = B_{\mathfrak{m}_i}/\mathfrak{m}_i B_{\mathfrak{m}_i}$. Let $V_{E_i} = E_i \otimes_B V_B$. The following lemmas are analogous to Lemma 3.5 and 3.7.

Lemma 3.15. *V_B has Galois type τ if and only if V_{E_i} has Galois type τ for each $i = 1, \dots, n$.*

Proof. It follows directly from Lemma 3.14. \square

Lemma 3.16. *Let E' be a finite extension of E , and let $B_{E'} = E' \otimes_E B$ and $V_{B_{E'}} = B_{E'} \otimes_B V_B$. Then V_B has Galois type τ if and only if $V_{B_{E'}}$ has Galois type τ .*

Proof. Since the natural map of E -algebras $B \rightarrow B_{E'}$ is injective, it follows from Lemma 3.14. \square

The following theorem is essential in studying the locus of representations with a given Galois type.

Theorem 3.17. *Let τ be a Galois type, and let L/K be a finite Galois extension in \bar{K} over which τ becomes trivial. Let A be a finite flat \mathcal{O}_E -algebra and $\rho : \mathcal{G}_K \rightarrow \mathrm{GL}_d(A)$ be a Galois representation such that $\rho \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is semi-stable over L having Hodge-Tate weights in $[0, r]$.*

Suppose that for each positive integer n , there exist a finite flat \mathcal{O}_E -algebra A_n , a Galois representation $\rho_n : \mathcal{G}_K \rightarrow \mathrm{GL}_d(A_n)$, and an \mathcal{O}_E -linear surjection $\beta_n : A_n \rightarrow A/p^n A$ such that $A/p^n A \otimes_A \rho \cong A/p^n A \otimes_{\beta_n, A_n} \rho_n$ as $A[\mathcal{G}_K]$ -modules, and that $\rho_n \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is semi-stable over L having Hodge-Tate weights in $[0, r]$ and Galois type τ .

Then $\rho \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ also has Galois type τ .

Proof. Let $B = A[\frac{1}{p}]$. We have $B_{\mathrm{red}} \cong \prod_i E_i$ for some finite extensions E_i/E . Let H/E be a finite Galois extension containing the Galois closures of E_i for all i . We write $A_{\mathcal{O}_H} = \mathcal{O}_H \otimes_{\mathcal{O}_E} A$. Then $(A_{\mathcal{O}_H}[\frac{1}{p}])_{\mathrm{red}} \cong H \otimes_E B_{\mathrm{red}} \cong H \otimes_E \prod_i E_i$. Since H contains the Galois closures of E_i for all i , $H \otimes_E E_i \cong \prod_j H$ with E_i embedding to H differently. This induces the natural map $\psi_l : A_{\mathcal{O}_H} \rightarrow A_{\mathcal{O}_H}[\frac{1}{p}] \rightarrow H$ to the l -th factor of $\prod_j H$. Let $A_l = \psi_l(A_{\mathcal{O}_H})$. Since $\psi_l : A_{\mathcal{O}_H} \rightarrow A_l \subset H$ is a morphism of \mathcal{O}_H -algebras, $A_l = \mathcal{O}_H$. By Lemma 3.15 and 3.16, it suffices to show that $H \otimes_{\psi_l, A_{\mathcal{O}_H}} (A_{\mathcal{O}_H} \otimes_A \rho)$ has Galois type τ . Therefore, we may and will replace A by \mathcal{O}_H , ρ by $\mathcal{O}_H \otimes_{\psi_l, A_{\mathcal{O}_H}} (A_{\mathcal{O}_H} \otimes_A \rho)$, A_n by $\mathcal{O}_H \otimes_{\mathcal{O}_E} A_n$, and replace β_n and ρ_n accordingly.

Denote by L_0 the maximal unramified extension of K_0 contained in L . Note that $I_{L/K} \cong I_{L/KL_0}$. Applying the results of Section 2 with (L_0, KL_0, L) in place of (K_0, K, K') , we get the associated lattice $M_{\mathrm{st}}(\rho) \in L^r(\varphi, N, \mathrm{Gal}(L/KL_0))$ in $D_{\mathrm{st}}^L(\rho \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$. By the proof of Lemma 3.11, $M_{\mathrm{st}}(\rho)$ is a free $\mathcal{O}_H \otimes_{\mathbf{Z}_p} \mathcal{O}_{L_0}$ -module of rank d . Thus, for all $\sigma \in I_{L/K}$, $\mathrm{tr}(\sigma | D_{\mathrm{st}}^L(\rho \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)) = \mathrm{tr}(\sigma | M_{\mathrm{st}}(\rho)) \in \mathcal{O}_H$.

Now, fix a positive integer n , and let $B_n = A_n[\frac{1}{p}]$. $B_{n, \mathrm{red}} \cong \prod_i F_i$ for some finite extensions F_i/E . Let H'/H be a finite Galois extension which contains the Galois closures of F_i for all i . Similarly as in the proof of Theorem 3.3, we see that $\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} A_n$ is good (as defined in Section 3.1). Thus, $M_{\mathrm{st}}(\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho_n) \in L^r(\varphi, N, \mathrm{Gal}(L/KL_0))$ is a free $(\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} A_n) \otimes_{\mathbf{Z}_p} \mathcal{O}_{L_0}$ -module of rank d by the proof of Lemma 3.11.

Note that we have the surjection $\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \beta_n : \mathcal{O}_{H'} \otimes_{\mathcal{O}_H} A_n \rightarrow \mathcal{O}_{H'} \otimes_{\mathcal{O}_H} A/p^n A = \mathcal{O}_{H'}/p^n \mathcal{O}_{H'}$. Denote by T the torsion \mathcal{G}_K -representation $\mathcal{O}_{H'}/p^n \mathcal{O}_{H'} \otimes_{\mathcal{O}_{H'}} (\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho) \cong \mathcal{O}_{H'}/p^n \mathcal{O}_{H'} \otimes_{\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \beta_n, \mathcal{O}_{H'} \otimes_{\mathcal{O}_H} A_n} (\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho_n)$. T has two lifts j_1 and j_2 corresponding to $\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho$ and $\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho_n$ respectively, and we obtain $M_{\mathrm{st}, j_1}(T), M_{\mathrm{st}, j_2}(T) \in M_{\mathrm{tor}}^{\mathrm{fil}, r}(\varphi, N, \mathrm{Gal}(L/KL_0))$. Note that $M_{\mathrm{st}, j_1}(T) \cong \mathcal{O}_{H'}/p^n \mathcal{O}_{H'} \otimes_{\mathcal{O}_{H'}} M_{\mathrm{st}}(\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho)$ and $M_{\mathrm{st}, j_2}(T) \cong (\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} A_n) / \ker(\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \beta_n) \otimes_{\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} A_n} M_{\mathrm{st}}(\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho_n)$. Thus, $M_{\mathrm{st}, j_1}(T)$ and $M_{\mathrm{st}, j_2}(T)$ are free over $\mathcal{O}_{H'}/p^n \mathcal{O}_{H'} \otimes_{\mathbf{Z}_p} \mathcal{O}_{L_0}$ of rank d , for which we fix a choice of bases. Let $\sigma \in I_{L/K}$, and for $i = 1, 2$, let C_i be the $d \times d$ matrix with coefficients in $\mathcal{O}_{H'}/p^n \mathcal{O}_{H'} \otimes_{\mathbf{Z}_p} \mathcal{O}_{L_0}$ which represents the σ -action on $M_{\mathrm{st}, j_i}(T)$ with respect to the chosen bases. By Corollary 2.8 and Proposition 2.9, there exist $I_{L/K}$ -equivariant $\mathcal{O}_{H'} \otimes_{\mathbf{Z}_p} \mathcal{O}_{L_0}$ -module morphisms $g_1 : M_{\mathrm{st}, j_1}(T) \rightarrow M_{\mathrm{st}, j_2}(T)$ and $g_2 : M_{\mathrm{st}, j_2}(T) \rightarrow M_{\mathrm{st}, j_1}(T)$ such that $g_1 \circ g_2 = p^{c''} \mathrm{Id}|_{M_{\mathrm{st}, j_2}(T)}$ and $g_2 \circ g_1 = p^{c''} \mathrm{Id}|_{M_{\mathrm{st}, j_1}(T)}$ where c'' is a constant depending only

on the Eisenstein polynomial for L/L_0 and r . For $i = 1, 2$, let D_i be the $d \times d$ matrix with coefficients in $\mathcal{O}_{H'}/p^n \mathcal{O}_{H'} \otimes_{\mathbf{Z}_p} \mathcal{O}_{L_0}$ representing g_i . Then $D_1 D_2 = D_2 D_1 = p^{c''} \text{Id}$ and $C_2 D_1 = D_1 C_1$. Thus,

$$\text{tr}(C_2 D_1 D_2) = \text{tr}(D_1 C_1 D_2) = \text{tr}(D_2 D_1 C_1),$$

i.e., $p^{c''} \text{tr}(C_1) = p^{c''} \text{tr}(C_2)$ in $\mathcal{O}_{H'}/p^n \mathcal{O}_{H'}$. Since $\text{tr}(\sigma|_{M_{\text{st}}(\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho_n)}) = \text{tr}(\tau(\sigma)) \in \mathcal{O}_E$ and $\text{tr}(\sigma|_{M_{\text{st}}(\mathcal{O}_{H'} \otimes_{\mathcal{O}_H} \rho)}) = \text{tr}(\sigma|_{M_{\text{st}}(\rho)}) \in \mathcal{O}_H$, we have

$$\text{tr}(\sigma|_{M_{\text{st}}(\tilde{\rho})}) - \text{tr}(\tau(\sigma)) \in p^{n-c''} \mathcal{O}_H.$$

Since this holds for all positive integers n , we have $\text{tr}(\sigma|_{M_{\text{st}}(\tilde{\rho})}) = \text{tr}(\tau(\sigma))$. \square

4 Galois Deformation Ring

We now construct the quotient of the universal deformation ring which corresponds to the locus of potentially semi-stable representations of a given Hodge-Tate type and Galois type. Let E/\mathbf{Q}_p be a finite extension with residue field \mathbf{F} . Denote by \mathcal{C} the category of topological local \mathcal{O}_E -algebras A satisfying the following conditions:

- The natural map $\mathcal{O}_E \rightarrow A/\mathfrak{m}_A$ is surjective.
- The map from A to the projective limit of its discrete artinian quotients is a topological isomorphism.

Note that the first condition implies \mathbf{F} is also the residue field of A . The second condition is equivalent to the condition that A is complete and its topology can be given by a collection of open ideals \mathfrak{a} for which A/\mathfrak{a} is artinian. Morphisms in \mathcal{C} are continuous \mathcal{O}_E -algebra homomorphisms.

Proposition 4.1. ([SL97, Proposition 2.4]) *Suppose A is a noetherian ring in \mathcal{C} . Then the topology on A is equal to the \mathfrak{m}_A -adic topology, and A is \mathfrak{m}_A -adically complete. Furthermore, every \mathcal{O}_E -algebra homomorphism $A \rightarrow A'$ with A' in \mathcal{C} is continuous.*

Let V_0 be a continuous \mathbf{F} -representation of \mathcal{G}_K having rank d . For $A \in \mathcal{C}$, a *deformation* of V_0 in A is an isomorphism class of continuous A -representations V of \mathcal{G}_K satisfying $\mathbf{F} \otimes_A V \cong V_0$ as $\mathbf{F}[\mathcal{G}_K]$ -modules. We denote by $\text{Def}(V_0, A)$ the set of such deformations. A morphism $A \rightarrow A'$ in \mathcal{C} induces a map $f_* : \text{Def}(V_0, A) \rightarrow \text{Def}(V_0, A')$ sending the class of a representation V over A to the class of $A' \otimes_{f,A} V$. Assume V_0 is absolutely irreducible. Then, the following is proved in [SL97].

Proposition 4.2. (cf. [SL97, Theorem 2.3]) *There exists a universal deformation ring $R \in \mathcal{C}$ and a deformation $V_R \in \text{Def}(V_0, R)$ such that for all rings $A \in \mathcal{C}$, we have a bijection*

$$\text{Hom}_{\mathcal{C}}(R, A) \xrightarrow{\cong} \text{Def}(V_0, A) \quad (4.1)$$

given by $f \mapsto f_(V_R)$. The ring R is noetherian if and only if $\dim_{\mathbf{F}} H^1(\mathcal{G}_K, \text{End}_{\mathbf{F}}(V_0))$ is finite.*

Note that if K/\mathbf{Q}_p is not finite, then R is not necessarily noetherian in general.

We fix a Hodge-Tate type \mathbf{v} and Galois type τ , and let L/K be a finite Galois extension over which τ becomes trivial. Let \mathcal{C}^0 be the full subcategory of \mathcal{C} consisting of artinian rings. Abusing the notation, we write $V \in \text{Def}(V_0, A)$ for a continuous A -representation V to mean that $\mathbf{F} \otimes_A V \cong V_0$. For $A \in \mathcal{C}^0$ and a \mathcal{G}_K -representation $V_A \in \text{Def}(V_0, A)$, we say V_A is *potentially semi-stable of type (\mathbf{v}, τ)* if there exist a finite flat \mathcal{O}_E -algebra B , a surjection $g : B \rightarrow A$ of \mathcal{O}_E -algebras, and a continuous B -representation V_B of \mathcal{G}_K such that $V_B \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is potentially semi-stable having Hodge-Tate type \mathbf{v} and Galois type τ , and $A \otimes_{g,B} V_B \cong V_A$ as $A[\mathcal{G}_K]$ -modules. For $A \in \mathcal{C}$, denote by $S_{\mathbf{v},\tau}(A)$ the subset of $\text{Def}(V_0, A)$ consisting of the isomorphism classes of representations V_A such that $A/\mathfrak{a} \otimes_A V_A$ is potentially semi-stable of type (\mathbf{v}, τ) for all open ideals $\mathfrak{a} \subsetneq A$.

Proposition 4.3. *For any \mathcal{C} -morphism $f : A \rightarrow A'$, we have $f_*(S_{\mathbf{v},\tau}(A)) \subset S_{\mathbf{v},\tau}(A')$. There exists a closed ideal $\mathfrak{a}_{\mathbf{v},\tau}$ of the universal deformation ring R such that the map (4.1) induces a bijection $\text{Hom}_{\mathcal{C}}(R/\mathfrak{a}_{\mathbf{v},\tau}, A) \xrightarrow{\cong} S_{\mathbf{v},\tau}(A)$.*

Proof. We check the conditions in [SL97, Section 6]. Let $f : A \hookrightarrow A'$ be an inclusion of artinian rings in \mathcal{C} , and let $V_A \in \text{Def}(V_0, A)$ be a representation. We first claim that $V_A \in S_{\mathbf{v},\tau}(A)$ if and only if $V_{A'} := A' \otimes_{f,A} V_A \in S_{\mathbf{v},\tau}(A')$. Suppose that $V_A \in S_{\mathbf{v},\tau}(A)$. Then there exist a finite flat \mathcal{O}_E -algebra B , a surjection $g : B \rightarrow A$, and a B -representation V_B such that $V_B \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is potentially semi-stable having Hodge-Tate type \mathbf{v} and Galois type τ , and $A \otimes_{g,B} V_B \cong V_A$. There exists a surjection $f' : A[x_1, \dots, x_n] \twoheadrightarrow A'$ of \mathcal{O}_E -algebras extending f such that $f'(x_i) \in \mathfrak{m}_{A'}$ for each i . Let $I_{m,A} \subset A[x_1, \dots, x_n]$ denote the ideal generated by the m -th degree homogeneous polynomials with coefficients in A . Since A' is artinian, $f'(I_{m,A}) = 0$ for a sufficiently large m , and f' induces a surjection $A[x_1, \dots, x_n]/I_{m,A} \twoheadrightarrow A'$ for such m . Thus, we have surjective homomorphisms of \mathcal{O}_E -algebras

$$g' : B' := B[x_1, \dots, x_n]/I_{m,B} \twoheadrightarrow A[x_1, \dots, x_n]/I_{m,A} \twoheadrightarrow A'.$$

Note that B' is a finite flat \mathcal{O}_E -algebra. Let $V_{B'} = B' \otimes_B V_B$. Then $V_{B'} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is semi-stable over L . By Lemma 3.2 and 3.14, it has Hodge-Tate type \mathbf{v} and Galois type τ . $A' \otimes_{g',B'} V_{B'} \cong V_{A'}$, so $V_{A'} \in S_{\mathbf{v},\tau}(A')$.

Conversely, suppose $V_{A'} \in S_{\mathbf{v},\tau}(A')$. Then there exist a finite flat \mathcal{O}_E -algebra B' , a surjection $g' : B' \rightarrow A'$, and a B' -representation $V_{B'}$ such that $V_{B'} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is potentially semi-stable having Hodge-Tate type \mathbf{v} and Galois type τ , and $A' \otimes_{g',B'} V_{B'} \cong V_{A'}$. Let B

be the kernel of the composite of morphisms $B' \xrightarrow{g'} A' \rightarrow A'/f(A)$. Then B is a finite flat \mathcal{O}_E -algebra, and we have the surjection $g : B \twoheadrightarrow A$ of \mathcal{O}_E -algebras induced from g' . Let V_B be the kernel of the composite

$$V_{B'} \rightarrow A' \otimes_{g', B'} V_{B'} \cong V_{A'} \rightarrow A'/f(A) \otimes_{A'} V_{A'}.$$

Then V_B is a continuous B -representation of \mathcal{G}_K such that $V_{B'} = B' \otimes_B V_B$ and $A \otimes_{g, B} V_B \cong V_A$. By the main theorem for semi-stable representations in [Liu07], $V_B \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is semi-stable over L . It has Hodge-Tate type \mathbf{v} and Galois type τ by Lemma 3.8 and 3.14, and therefore $V_A \in S_{\mathbf{v}, \tau}(A)$.

Now, for $A \in \mathcal{C}$ and a representation $V_A \in \text{Def}(V_0, A)$, suppose $\mathfrak{a}_1, \mathfrak{a}_2 \subsetneq A$ are open ideals such that $A/\mathfrak{a}_i \otimes_A V_A \in S_{\mathbf{v}, \tau}(A/\mathfrak{a}_i)$ for $i = 1, 2$. We claim that $A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \otimes_A V_A \in S_{\mathbf{v}, \tau}(A/(\mathfrak{a}_1 \cap \mathfrak{a}_2))$. There exist a finite flat \mathcal{O}_E -algebra B_i , a surjection $g_i : B_i \twoheadrightarrow A/\mathfrak{a}_i$, and a B_i -representation V_{B_i} such that $V_{B_i} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is potentially semi-stable having Hodge-Tate type \mathbf{v} and Galois type τ , and that $A/\mathfrak{a}_i \otimes_{g_i, B_i} V_{B_i} \cong A/\mathfrak{a}_i \otimes_A V_A$. Let $V_{B_1 \times B_2}$ be the $(B_1 \times B_2)$ -representation corresponding to $V_{B_1} \oplus V_{B_2}$. Note that $V_{B_1 \times B_2}$ is potentially semi-stable having Hodge-Tate type \mathbf{v} and Galois type τ . Consider the natural inclusion $A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \subset A/\mathfrak{a}_1 \times A/\mathfrak{a}_2$. Let B be the kernel of the composite of morphisms

$$B_1 \times B_2 \xrightarrow{g_1 \times g_2} A/\mathfrak{a}_1 \times A/\mathfrak{a}_2 \rightarrow (A/\mathfrak{a}_1 \times A/\mathfrak{a}_2)/(A/(\mathfrak{a}_1 \cap \mathfrak{a}_2)).$$

Then B is a finite flat \mathcal{O}_E -algebra, and we have the surjection $g : B \twoheadrightarrow A/(\mathfrak{a}_1 \cap \mathfrak{a}_2)$ induced from $g_1 \times g_2$. Let V_B be the kernel of the composite of morphisms

$$V_{B_1 \times B_2} \rightarrow (A/\mathfrak{a}_1 \times A/\mathfrak{a}_2) \otimes_{g_1 \times g_2, B_1 \times B_2} V_{B_1 \times B_2} \cong (A/\mathfrak{a}_1 \times A/\mathfrak{a}_2) \otimes_A V_A$$

and

$$(A/\mathfrak{a}_1 \times A/\mathfrak{a}_2) \otimes_A V_A \rightarrow (A/\mathfrak{a}_1 \times A/\mathfrak{a}_2)/(A/(\mathfrak{a}_1 \cap \mathfrak{a}_2)) \otimes_A V_A.$$

Then V_B is a continuous B -representation of \mathcal{G}_K such that $(B_1 \times B_2) \otimes_B V_B \cong V_{B_1 \times B_2}$ and $A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \otimes_{g, B} V_B \cong A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \otimes_A V_A$. By the main theorem for semi-stable representations in [Liu07], $V_B \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is semi-stable over L . It has Hodge-Tate type \mathbf{v} and Galois type τ by Lemma 3.8 and 3.14. Thus, $A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \otimes_A V_A \in S_{\mathbf{v}, \tau}(A/(\mathfrak{a}_1 \cap \mathfrak{a}_2))$.

The result then follows by [SL97, Proposition 6.1]. \square

Finally, we prove the main theorem.

Theorem 4.4. *Let A be a finite flat \mathcal{O}_E -algebra, and let $f : R \rightarrow A$ be a continuous \mathcal{O}_E -algebra homomorphism (where we equip A with the (p) -adic topology). Then the induced representation $A[\frac{1}{p}] \otimes_{f, R} V_R$ is potentially semi-stable of Hodge-Tate type \mathbf{v} and Galois type τ if and only if f factors through the quotient $R/\mathfrak{a}_{\mathbf{v}, \tau}$.*

Proof. Let $A_1 = f(R) \subset A$. Then A_1 is a finite flat \mathcal{O}_E -algebra and local. We equip A_1 with the (p) -adic topology. Then $A_1 \in \mathcal{C}$, and the map $f : A \rightarrow A_1$ is a morphism in \mathcal{C} . Let $V_{A_1} = A_1 \otimes_{f,R} V_R$ and $V_A = A \otimes_{f,R} V_R \cong A \otimes_{A_1} V_{A_1}$.

Suppose that $V_A \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is potentially semi-stable of Hodge-Tate type \mathbf{v} and Galois type τ . By the main theorem for semi-stable representations in [Liu07], $V_{A_1} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is semi-stable over L . By Lemma 3.8 and 3.14, $V_{A_1} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ has Hodge-Tate type \mathbf{v} and Galois type τ . Thus, $V_{A_1} \in S_{\mathbf{v},\tau}(A_1)$, and f factors through $R/\mathfrak{a}_{\mathbf{v},\tau}$ by Proposition 4.3.

Conversely, suppose f factors through $R/\mathfrak{a}_{\mathbf{v},\tau}$. Then $V_{A_1} \in S_{\mathbf{v},\tau}(A_1)$ by Proposition 4.3, so $A_1/p^n \otimes_{A_1} V_{A_1}$ is potentially semi-stable of type (\mathbf{v}, τ) for each $n \geq 1$. By the main theorem for semi-stable representations in [Liu07], $V_{A_1} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is semi-stable over L . And by Theorem 3.3 and 3.17, $V_{A_1} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ has Hodge-Tate type \mathbf{v} and Galois type τ . Thus, $V_A \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is potentially semi-stable of Hodge-Tate type \mathbf{v} and Galois type τ . \square

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